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MEAN VALUE OF THE TWISTED COCHRANE SUMS

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Abstract

Let q be an odd prime and χ be the non-principal Dirichlet character mod q. In this paper, the authors studied the mean value of the classical Cochrane sums twisted by χ and gave an asymptotic formula for it.

1. Introduction

Cochrane introduced a sum such as

$$C(h,\, q) = \sum_{a=1}^{q}{}' \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{\overline{a}h}{q}\right)\right),$$

which is analogous to Dedekind sums and we prefer to call Cochrane sums. Here q is a positive integer and h is an arbitrary integer, $a\overline{a} \equiv 1$ (mod q),

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$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

and $\sum'$ denotes the summation over all a with $1 \leqslant a \leqslant q$, $(a, q) = 1$.

Various properties of Cochrane sums have been investigated by many scholars. For example, Zhang and Yi [6] obtained the upper bound as

$$C(h, q) \ll q^{1/2} \tau(q) \log^2 q,$$

where $\tau(q)$ is the Dirichlet divisor function. In [7], Zhang studied the mean square value and obtained for $q \geqslant 2$,

$$\sum_{h=1}^{q} C^2(h, q) = \frac{5}{144} \varphi^2(q) \prod_{p^\alpha \| q} \left(\frac{(p+1)^2}{p^2+1} + \frac{1}{p^{3\alpha}} \right) \left(1 + \frac{1}{p} + \frac{1}{p^2} \right)^{-1} + O(q^{1+o(1)}), \tag{1}$$

where $\varphi(q)$ is the Euler function. Zhang [8, 9] also studied the mean value weighted by Kloosterman sums. Furthermore, Lu and Yi [1] gave an asymptotic formula for the mean square value over short intervals.

It is quite natural to consider the corresponding sums twisted by Dirichlet characters. Suppose χ is a Dirichlet character mod q, the twisted Cochrane sum is defined by

$$C_\chi(h, q) = \sum_{a \bmod q}{}' \chi(a) \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{\bar{a} h}{q} \right) \right). \tag{2}$$

It is clear that $((-x)) = -((x))$, thus $C_\chi(h, q) = 0$, if $\chi(-1) = -1$. Quite recently, Ren and Yi [3] obtained an asymptotic formula for the mean square value as $q \equiv 1 \pmod 4$ being a prime and χ being the Legendre symbol mod q. This present note focuses on the mean square value

$$\sum_{h=1}^{q} |C_\chi(h, q)|^2,$$

for an arbitrary character χ mod q with $\chi(-1) = 1$.

Theorem 1. *Let q be an odd prime and χ be an arbitrary non-principal character mod q with $\chi(-1) = 1$. We have*

$$\sum_{h=1}^{q} |C_\chi(h, q)|^2 = \frac{1}{2\pi^4} q^2 c(\chi, q) + O(q \log^4 q),$$

where the O-constant is absolute and

$$c(\chi, q) = \zeta(2)|L(2, \chi)|^2 \prod_{\chi(p)=1} \left[\left(1 - \frac{1}{p^2}\right)^{-2} + 2p^{-4}\left(1 - \frac{1}{p^2}\right)^{-3}\right]$$

$$\times \prod_{\substack{(p, q)=1 \\ \chi(p)\neq 1}} \left[\left(1 - \frac{\chi(p)}{p^2}\right)(1 - \chi(p))^{-1} + \left(1 - \frac{\overline{\chi}(p)}{p^2}\right)(1 - \overline{\chi}(p))^{-1}\right],$$

where $\zeta(s)$ and $L(s, \chi)$ denote the Riemann zeta function and Dirichlet L-function, respectively.

2. Proof of the Theorem

First, we introduce the classical Kloosterman sums. Let m, n, and c be integers, $c \geqslant 1$, the classical Kloosterman sum is defined by

$$S(m, n; c) = \sum_{a \bmod c}{}' e\left(\frac{ma + n\overline{a}}{c}\right),$$

where $e(x) = e^{2\pi i x}$. The well-known estimate essentially due to Weil [5] is

$$|S(m, n; c)| \leqslant c^{1/2}(m, n, c)^{1/2}\tau(c),$$

where (m, n, c) is the greatest common divisor of m, n, and c. The twisted Kloosterman sum is defined by

$$S_\chi(m, n; c) = \sideset{}{'}\sum_{a \bmod c} \chi(a) e\left(\frac{ma + n\bar{a}}{c}\right),$$

which enjoys the same upper bound

$$|S_\chi(m, n; c)| \leqslant c^{1/2}(m, n, c)^{1/2}\tau(c). \tag{3}$$

The estimate (3) plays a quite important role throughout this paper.

Now, we consider the mean square value

$$\sum_{h=1}^{q} |C_\chi(h, q)|^2,$$

for an odd prime q and χ mod q with $\chi(-1) = 1$. By introducing the Fourier expansion,

$$((x)) = -\frac{1}{2\pi i} \sum_{0<|n|\leqslant N} \frac{e(nx)}{n} + O(N^{-1}), \text{ for } x \in \mathbb{R},\ N > 0,$$

we have

$$\begin{aligned} C_\chi(h, q) &= -\frac{1}{4\pi^2} \sum_{0<|m|\leqslant N} \sum_{0<|n|\leqslant N} \frac{S_\chi(m, hn; q)}{mn} + O(qN^{-1}) \\ &= -\frac{1}{4\pi^2} \sideset{}{'}\sum_{0<|m|\leqslant N} \sideset{}{'}\sum_{0<|n|\leqslant N} \frac{S_\chi(m, hn; q)}{mn} \\ &\quad + O(q^{-1/2}\log^2 N + qN^{-1}). \end{aligned}$$

Thus from (3), we have

$$\begin{aligned} &\sum_{h=1}^{q} |C_\chi(h, q)|^2 \\ &= \frac{1}{16\pi^4} \sum_{h=1}^{q} \left| \sideset{}{'}\sum_{0<|m|\leqslant N} \sideset{}{'}\sum_{0<|n|\leqslant N} \frac{S_\chi(m, hn; q)}{mn} + O(q^{-1/2}\log^2 N + qN^{-1}) \right|^2 \\ &= \frac{1}{16\pi^4} W(h, q) + O\left(q\log^4 N + q^2N^{-2} + N^{-1}q^{5/2}\log^2 N\right), \end{aligned}$$

where

$$W(h,\ q) = \sum_{h=1}^{q}\left|\sum_{0<|m|\leqslant N}{}' \sum_{0<|n|\leqslant N}{}' \frac{S_\chi(m,\ hn;\ q)}{mn}\right|^2.$$

Opening the square in $W(h,\ q)$, we have

$$W(h,\ q) = \sum_{\substack{0<|m_1|,|m_2|,|n_1|,|n_2|\leqslant N\\ (m_1m_2n_1n_2,\ q)=1}}\sum\sum\sum \frac{1}{m_1m_2n_1n_2}\sum_{h=1}^{q} S_\chi(m_1,\ hn_1;\ q)\overline{S_\chi(m_2,\ hn_2;\ q)}.$$

For $(m_1m_2n_1n_2,\ q) = 1,$ we have

$$\sum_{h=1}^{q} S_\chi(m_1,\ hn_1;\ q)\overline{S_\chi(m_2,\ hn_2;\ q)}$$

$$= \sum_{a \bmod q}{}' \sum_{b \bmod q}{}' \chi(a\bar{b})e\left(\frac{m_1a - m_2b}{q}\right)\sum_{h=1}^{q} e\left(\frac{h(n_1\bar{a} - n_2\bar{b})}{q}\right)$$

$$= q \sum_{\substack{a \bmod q\\ n_1\bar{a}\equiv n_2\bar{b} (\bmod q)}}{}' \sum_{b \bmod q}{}' \chi(a\bar{b})e\left(\frac{m_1a - m_2b}{q}\right)$$

$$= \chi(n_1\overline{n_2})q^2,$$

if $m_1n_1 \equiv m_2n_2 (\bmod q),$ and it vanishes otherwise. Hence

$$W(h,\ q) = q^2 \sum_{\substack{0<|m_1|,|m_2|,|n_1|,|n_2|\leqslant N\\ m_1n_1\equiv m_2n_2 (\bmod q)\\ (m_1m_2n_1n_2,\ q)=1}}\sum\sum\sum \frac{\chi(n_1\overline{n_2})}{m_1m_2n_1n_2}.$$

Now, we separate the summations over $m_1,\ m_2,\ n_1,$ and n_2 into two parts, namely,

$$W(h,\ q) = q^2\left(\sum_{m_1n_1=m_2n_2}\sum\sum\sum + \sum_{m_1n_1\neq m_2n_2}\sum\sum\sum\right).$$

The terms with $m_1n_1 \neq m_2n_2$ contributes to $W(h, q)$ at most

$$2q^2\left|\sum\sum\sum\sum_{\substack{0<|m_1|,|m_2|,|n_1|,|n_2|\leqslant N\\ q|m_1n_1-m_2n_2>0\\ (m_1m_2n_1n_2,q)=1}}\frac{\chi(n_1\overline{n_2})}{m_1m_2n_1n_2}\right|$$

$$\ll q^2\sum_{m_1\leqslant N}\frac{1}{m_1}\sum_{k\leqslant N/q}\left|\sum_{n_1\leqslant N}\frac{\chi(n_1)}{n_1(m_1n_1+kq)}\right|\left|\sum_{n_2\leqslant N}\chi(n_2)\right|$$

$$\ll q\log^4 N,$$

where we have used Abel's summation formula and the Pólya-Vinogradov bound (see [2, 4])

$$\max_{M,N}\left|\sum_{M<n\leqslant M+N}\chi(n)\right|\ll q^{1/2}\log q.$$

Therefore,

$$W(h, q) = q^2\sum\sum\sum\sum_{\substack{0<|m_1|,|m_2|,|n_1|,|n_2|\leqslant N\\ m_1n_1=m_2n_2\\ (m_1m_2n_1n_2,q)=1}}\frac{\chi(n_1\overline{n_2})}{m_1m_2n_1n_2}+O(q\log^4 N).$$

Since $\chi(-1)=1$, thus

$$W(h, q) = 8q^2\sum\sum\sum\sum_{\substack{1\leqslant m_1,m_2,n_1,n_2\leqslant N\\ m_1n_1=m_2n_2\\ (m_1m_2n_1n_2,q)=1}}\frac{\chi(n_1\overline{n_2})}{m_1m_2n_1n_2}+O(q\log^4 N)$$

$$= 8q^2\sum_{n\leqslant N^2}{}'\frac{1}{n^2}\left|\sum_{\substack{d|n\\ d,n/d\leqslant N}}\chi(d)\right|^2+O(q\log^4 N)$$

$$= 8q^2 \sideset{}{'}\sum_{n \leqslant N} \frac{1}{n^2} \left| \sum_{\substack{d|n \\ d,\, n/d \leqslant N}} \chi(d) \right|^2 + 8q^2 \sideset{}{'}\sum_{N < n \leqslant N^2} \frac{1}{n^2} \left| \sum_{\substack{d|n \\ d,\, n/d \leqslant N}} \chi(d) \right|^2 + O(q \log^4 N)$$

$$= 8q^2 \sideset{}{'}\sum_{n \leqslant N} \frac{1}{n^2} \left| \sum_{d|n} \chi(d) \right|^2 + O\left(q^2 \sideset{}{'}\sum_{n > N} \frac{\tau^2(n)}{n^2} \right) + O(q \log^4 N)$$

$$= 8q^2 \sideset{}{'}\sum_{n \geqslant 1} \frac{1}{n^2} \left| \sum_{d|n} \chi(d) \right|^2 + O\left(q^2 \sideset{}{'}\sum_{n > N} \frac{\tau^2(n)}{n^2} \right) + O(q \log^4 N)$$

$$= 8q^2 A(\chi, q) + O(N^{-1} q^2 \log^3 N + q \log^4 N),$$

where

$$A(\chi, q) = \sideset{}{'}\sum_{n \geqslant 1} \frac{1}{n^2} \left| \sum_{d|n} \chi(d) \right|^2,$$

is a constant depending only on q and χ.

Hence

$$\sum_{h=1}^{q} |C_\chi(h, q)|^2 = \frac{1}{2\pi^4} q^2 A(\chi, q) + O(q \log^4 q), \tag{4}$$

by taking $N = q^3$.

Now it suffices to compute $A(\chi, q)$. Suppose χ is a d-th $(d > 1)$ character mod q, that is, $\chi^d(a) = 1$ for each a with $(a, q) = 1$, we obtain from the Euler product and the fact $1 + \chi(a) + \cdots + \chi^{d-1}(a) = 0$ for all a with $(a, q) = 1$, $\chi(a) \neq 1$ that

$$
\begin{aligned}
A(\chi,\, q) &= \prod_{(p,q)=1} \sum_{k \geqslant 0} \frac{1}{p^{2k}} \left| \sum_{s | p^k} \chi(s) \right|^2 \\
&= \prod_{(p,q)=1} \sum_{k \geqslant 0} \frac{1}{p^{2k}} \left| 1 + \chi(p) + \cdots + \chi^k(p) \right|^2 \\
&= B(\chi,\, q) \prod_{\chi(p)=1} \sum_{k \geqslant 0} \frac{(k+1)^2}{p^{2k}} \\
&= B(\chi,\, q) \prod_{\chi(p)=1} \left[\left(1 - \frac{1}{p^2}\right)^{-2} + 2p^{-4} \left(1 - \frac{1}{p^2}\right)^{-3} \right],
\end{aligned}
$$

where

$$
\begin{aligned}
B(\chi,\, q) &= \prod_{\substack{(p,q)=1 \\ \chi(p)\neq 1}} \sum_{k \geqslant 0} \frac{1}{p^{2k}} \left| 1 + \chi(p) + \cdots + \chi^k(p) \right|^2 \\
&= \prod_{\substack{(p,q)=1 \\ \chi(p)\neq 1}} \sum_{\ell=0}^{d-1} \sum_{\substack{k \geqslant 0 \\ k \equiv \ell (\mathrm{mod}\, d)}} \frac{1}{p^{2k}} \left| 1 + \chi(p) + \cdots + \chi^{\ell}(p) \right|^2 \\
&= \prod_{\substack{(p,q)=1 \\ \chi(p)\neq 1}} \sum_{\ell=0}^{d-1} \sum_{k \geqslant 0} \frac{1}{p^{2(\ell+kd)}} \left| 1 + \chi(p) + \cdots + \chi^{\ell}(p) \right|^2 \\
&= \prod_{\substack{(p,q)=1 \\ \chi(p)\neq 1}} \left(1 - \frac{1}{p^{2d}}\right)^{-1} \sum_{\ell=0}^{d-1} \frac{\left| 1 + \chi(p) + \cdots + \chi^{\ell}(p) \right|^2}{p^{2\ell}} \\
&= \zeta(2) |L(2,\, \chi)|^2 \prod_{\substack{(p,q)=1 \\ \chi(p)\neq 1}} \left[\left(1 - \frac{\chi(p)}{p^2}\right) (1 - \chi(p))^{-1} \right. \\
&\quad \left. + \left(1 - \frac{\overline{\chi}(p)}{p^2}\right) (1 - \overline{\chi}(p))^{-1} \right] + O(q^{-2}),
\end{aligned}
$$

where $\zeta(s)$ and $L(s, \chi)$ denote the Riemann zeta function and Dirichlet L-function, respectively. Hence

$$A(\chi, q) = \zeta(2)|L(2, \chi)|^2 \prod_{\substack{(p, q)=1 \\ \chi(p)\neq 1}} \left[\left(1 - \frac{\chi(p)}{p^2}\right)(1 - \chi(p))^{-1} + \left(1 - \frac{\overline{\chi}(p)}{p^2}\right)(1 - \overline{\chi}(p))^{-1}\right]$$

$$\times \prod_{\chi(p)=1} \left[\left(1 - \frac{1}{p^2}\right)^{-2} + 2p^{-4}\left(1 - \frac{1}{p^2}\right)^{-3}\right] + O(q^{-2}).$$

The theorem follows immediately by inserting the expression of $A(\chi, q)$ to (4).

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